three warm-up examples

1) raymond queneau's *cent mille milliards de poems* (1961) – a book of 10 sonnets (14 lines each) cut horizontally between lines.

- this gives us $\boxed{\quad}$ choices for the first line, $\boxed{\quad}$ for the second line, etc. → giving us a total of $\boxed{\quad}$ possible realizations.

2) earle brown's *25 pages for 1-25 pianos* (1953) – 25 pages played in any order and either rightside-up or upside-down.

- unlike queneau's book, after the $\boxed{\quad}$ choices for the first page, we only have $\boxed{\quad}$ choices for the second page.
- this continues until all 25 pages are chosen, modeled mathematically as $\boxed{\quad}$ or 25 factorial.
- but wait! each page can be played one of two ways, thus we must also multiply by $\boxed{\quad}$ to account for an extra choice per page.
- this gives us $(25!)(2^{25})$ = 520 46984 26366 66622 69308 10880 00000 possible orderings of brown's *25 pages*.

3) stéphane mallarmé's *le livre* – an unfinished book of unbounded pages from which any number may be chosen in any order.

- this is slightly more complex than brown's work for two reasons: 1) we don't have to use all the pages, and 2) we don't know how many pages there are from which to choose.
- so, let's call the number of pages x. from brown's problem, we know that choosing all pages would give us $\boxed{\quad}$ possibilities.
- but what if we only wanted to read one page? or only two pages? clearly, we need to add up all possible readings at all possible lengths.
- choices for one page → $\boxed{\quad}$ choices for two pages → $\boxed{\quad}$ choices for three pages → $\boxed{\quad}$ choices for x pages → $\boxed{\quad}$
- now, let's add them up using a summation from the shortest length (1) to the longest length (x): $\sum_{n=1}^{x} \frac{x!}{(x-n)!}$
- assuming mallarmé finished with just 50 pages,

$$\sum_{n=1}^{50} \frac{50!}{(50-n)!} = 82674\ 07687\ 92772\ 58572\ 49658\ 10093\ 01773\ 30298\ 44864\ 49338\ 75630\ 08252\ 98500$$

morton feldman’s *durations II*

consists of 50 cello sonorities and 49 piano sonorities with the following directions: “duration of each sound is chosen by the performer.” so, how many possible orderings of sonorities are there?

let’s simplify the problem and go from there. let (x,y) denote the number of sonorities to be performed by the cello (x) and the piano (y).

case (x,1)	(1,1)	(2,1)	(3,1)	(4,1)
possibilities				

case (x,2)	(1,2)	(2,2)	(3,2)	(4,2)
possibilities			25	41

now, let’s plot what we have so far (in addition to cases (x,0), (x,3) and (x,4):

(x,y)	**0**	**1**	**2**	**3**	**4**
0	1	1	1	1	1
1	1	3	5	7	9
2	1	5	13	25	41
3	1	7	25	63	129
4	1	9	41	129	321

modeling (x,0) is a breeze → $d_0(y) = 1$

modeling (x,1) is easy, too → $d_1(y) = \boxed{}$

modeling (x,2) isn’t so easy → $d_2(y) = 2x^2 + 2x + 1$

modeling (x,3) is tough → $d_3(y) = \frac{4}{3}x^3 + 2x^2 + \frac{8}{3}x + 1$

a different degreed polynomial for each row – just like pascal’s triangle!

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

pascal’s triangle

$$P(x,y) = \binom{x}{y} = \frac{x!}{y!(x-y)!}$$

1
1 1
1 3 1
1 5 5 1
1 7 13 7 1
1 9 25 25 9 1

our numbers (delannoy numbers)

$$d(x,y) = \sum_{n=0}^{x} \binom{x}{n}\binom{y}{n} 2^n$$

thus, the number of realizations of feldman’s *durations II* is

$$d(49,50) = \sum_{n=0}^{49} \binom{49}{n}\binom{50}{n} 2^n = 63\ 28374\ 50542\ 24278\ 20349\ 52730\ 74482\ 24645$$

karlheinz stockhausen's *klavierstück XI*

consists of 19 musical fragments scattered across a large sheet of paper with the following rules (abb.):

- randomly glance at any fragment and play it;
- randomly glace at *another* fragment and play it;
- continue until a fragment is reached for the third time, at which point the realization ends.

so, how many possible realizations (orderings of fragments) are there?

depending on the interpretation of the second rule, we have two possible answers:

- interpretation 1—the performer **may not** perform a fragment twice in a row.
- interpretation 2—the performer **may** perform a fragment twice in a row.

interpretation 1 was solved by Ronald Read and Lily Yen in 1996, so let's solve interpretation 2.

to enumerate all possible realizations, let's divide up the larger problem into 19 smaller ones:

- case 1—these realizations include exactly *one* repeated fragment
- case 2—these realizations include exactly *two* repeated fragments

  … and so on, up to
- case 19—these realizations include exactly *nineteen* repeated fragments

if we counted all the possibilities for each case, we could add them together, giving us our total!

[case 1] clearly, the shortest realization in this case is of length $\boxed{}$ while the longest realization is length $\boxed{}$. For each realization length in case 1, we must consider:

- how many fragments of the nineteen are used, and
- how many different arrangements the paired fragments can take (since they're identical and can't be distinguished by the listener).

let's construct a table counting all possible case 1 realizations at each length (let A be our paired fragment and x be any unpaired fragments):

case 1 length	example	fragment choices	number of arrangements	number of realizations
2	AA	19	1	(19)(1)
3	xAA	(19)(18)	3	(19)(18)(3)
4	AxAx			
5	xAAxx	(19)(18)(17)(16)	10	(19)(18)(17)(16)(10)
$2 \le n \le 20$		$\frac{19!}{(20-n)!}$	$\binom{n}{2}$	$\frac{19!}{(20-n)!}\binom{n}{2}$

therefore, counting up all lengths of case 1 realizations (letting n go from 2 to 20), we get

$$\sum_{n=2}^{20} \frac{19!}{(20-n)!}\binom{n}{2} = 56709\ 16171\ 25522\ 86009$$

[case 2] the shortest length case 2 realization is $\square$; the longest length realization is $\square$.

how many arrangements does our shortest length realization have? $\square$ (this will be important.)

case 2 length	example	fragment choices	number of arrangements	number of realizations
4	ABBA	(19)(18)	3	(19)(18)(3)
5	ABAxB	(19)(18)(17)	(3)(5)	(19)(18)(17)(3)(5)
6	AxABxB	(19)(18)(17)(16)		
$4 \le n \le 21$		$\dfrac{19!}{(21-n)!}$		

summing the realizations at every length of case 2 gives us

$$\sum_{n=4}^{21}(3)\frac{19!}{(21-n)!}\binom{n}{4} = 48\ 88106\ 54029\ 74824\ 31276$$

[case 3] shortest length realization → $\square$ longest length realization → $\square$

arrangements of shortest length realization → $\square$ (notice a pattern yet?)

case 3 length	example	fragment choices	number of arrangements	number of realizations
6	ABACCB	(19)(18)(17)	15	(19)(18)(17)(15)
7	ABxCBCA	(19)(18)(17)(16)	(15)(7)	(19)(18)(17)(16)(15)(7)
8	xAABCxCB	(19)(18)(17)(16)(15)	(15)(28)	(19)(18)(17)(16)(15)(15)(28)
$6 \le n \le 22$				

therefore, there are $\displaystyle\sum_{n=6}^{22}(15)\frac{19!}{(22-n)!}\binom{n}{6}$ = 2803 94075 65650 88172 01930 case 3 realizations.

enough case by case study—let's deduce patterns and sum up cases 1-19. it looks like each case is

$$\sum_{x=\min length}^{\max length}(\#base\ forms)\frac{19!}{(\max length-n)!}\binom{n}{\min length}$$

to sum up all cases at once, we must relate everything to the common variable *c* (for case).

clearly, min *length* can be written as $\square$ and max *length* as $\square$. base forms are a bit harder:

	case 1	case 2	case 3	case *c*
number of base forms	1	3	15	

now we can rewrite our general equation summing all cases 1 to 19, giving us our answer:

$$\sum_{c=1}^{19}\left\{\sum_{n=2c}^{c+19}\left(\prod_{y=1}^{c}(2y-1)\right)\frac{19!}{(c+19-n)!}\binom{n}{2c}\right\} = 27486\ 29142\ 90496\ 43057\ 37593\ 04217\ 99385\ 84355$$